40. The fourth power of the common difference of an arithmetic progression with integer entris is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer. **[IIT - 2000]**

41. Let a_1, a_2,........ be positive real numbers in geometric progression. For each n, let A_n, G_n, H_n be respectively, the arithmetic mean, geometric mean, and harmonic mean of a_1, a_2,........., a_n. Find an expression for the geometric mean of G_1, G_2, ..., G_n in terms of A_1, A_2,.........,A_n, H_1, H_2,.........,H_n. **[IIT - 2001]**

42. Let a, b be positive real numbers. If a, A_1, A_2,b are in arithmetic progression, a, G_1, G_2, be are in geometric progression and a, H_1, H_2, b are in harmonic progression, show that

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2} = \frac{(2a+b)(a+2b)}{9ab}.$$

[IIT - 2002]

43. If a, b, c are in A.P., a^2, b^2, c^2 are in H.P., then prove that either $a = b = c$ or a, b, $-\frac{c}{2}$ form a G.P. **[IIT - 2003]**

44. If a, b, c are positive real numbers, then prove that :
$\{(1+a)(1+b)(1+c)\}^7 > 7^7 a^4b^4c^4$ **[IIT - 2004]**

45. A cricketer plays n(> 1) matches and scores $k(2^{n+1-k})$ runs in his kth match ($1 \le k \le n$). If the total rns scored by him in the n matches is $\frac{1}{4}(n+1)(2^{n+1}-n-2)$, find the value of n. **[IIT - 2005]**

46. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ...(-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then find the least natural number n_0 such that $b_n > a_n$ $\forall n \ge n_0$. **[IIT - 2006]**

E. Comperehension # 1

Let V_r denote the sum of first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is (2r – 1). Let $T_r = V_{r+1} - V_r - 2$ and $Q_t = T_{r+1} - T_r$ for r = 1, 2,.....

47. The sum $V_1 + V_2 + ... + V_n$ is **[IIT - 2007]**
(A) $\frac{1}{12}n(n+1)(3n^2-n+1)$
(B) $\frac{1}{12}n(n+1)(3n^2+n+2)$
(C) $1/2n(2n^2-n+1)$
(D) $1/3(2n^3-2n+3)$

48. T_r is always **[IIT - 2007]**
(A) an odd number
(B) an even number
(C) a prime number
(D) a composite number

49. Which one of the following is a correct statement? **[IIT - 2007]**
(A) Q_1, Q_2, Q_3,.... are in A.P., with common difference 5
(B) Q_1, Q_2, Q_3,.... are in A.P., with common difference 6
(C) Q_1, Q_2, Q_3,.... are in A.P., with common difference 11
(D) $Q_1 = Q_2 = Q_3 =$....

Comprehension # 2

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \ge 2$, Let A_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively.

28. Let the three digit numbers A28, 3B9, and 62C, where A, B, and C are integers between 0 and 9, be divisible by a fixed integer k. Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k.

[IIT - 1990]

29. If $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$ **[IIT - 1991]**

30. For a fixed positive integer n, if

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$ then show that

$\left[\frac{D}{(n!)^3} - 4\right]$ is divisible by n **[IIT - 1992]**

31. Let λ and α be real. Find the set of all values of λ for which the system of linear equations

$\lambda x + (\sin\alpha) y + (\cos\alpha) z = 0$

$x + (\cos\alpha) y + (\sin\alpha) z = 0$

$-x + (\sin\alpha) y - (\cos\alpha) z = 0$

has a non-trivial solution. For $\lambda = 1$, find all values of α. **[IIT - 1993]**

32. For all values of A, B, C & P, Q, R show that

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0.$$

[IIT - 1994]

33. Let $a > 0, d > 0$. Find the value of the determinant

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

[IIT - 1996]

34. Find the value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

where a, b and c are respectively the pth, qth and rth terms of harmonic progression.

[IIT - 1997]

35. Let a, b, c, d be real numbers in G.P. If u, v, w satisfy the system of equations, **[IIT - 1998]**

$u + 2v + 3w = 6$

$4u + 5v + 6w = 12$

$6u + 9v = 4$

then show that the roots of the equation

$\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right) x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2] x + u + v + w = 0$ and $20x^2 + 10(a-d)^2 x - 9 = 0$ are reciprocals of each other.

[IIT - 1999]

36. Prove that for all values of θ;

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin(\theta + 2\pi/3) & \cos(\theta + 2\pi/3) & \sin(2\theta + 4\pi/3) \\ \sin(\theta - 2\pi/3) & \cos(\theta - 2\pi/3) & \sin(2\theta - 4\pi/3) \end{vmatrix}$$

$= 0$ **[IIT - 2000]**

37. Let a, b, c be real no with $a^2 + b^2 + c^2 = 1$ then show that

$$\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix}$$

represent straight line **[IIT - 2001]**

$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix}$$

14. Evaluate the determinant

$$\Delta = \begin{vmatrix} \sqrt{p}+\sqrt{q} & 2\sqrt{r} & \sqrt{r} \\ \sqrt{qr}+\sqrt{2p} & r & \sqrt{2r} \\ q+\sqrt{pr} & \sqrt{qr} & r \end{vmatrix}$$

where p, q and r are positive real numbers.

15. If $a \neq p$, $b \neq q$, $c \neq r$ and

$$\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0,$$

then find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$

16. If a and x are real numbers and n is a positive integer prove that

$$\begin{vmatrix} a^n-x & a^{n+1}-x & a^{n+2}-x \\ a^{n+3}-x & a^{n+4}-x & a^{n+5}-x \\ a^{n+6}-x & a^{n+7}-x & a^{n+8}-x \end{vmatrix} = 0$$

17. Show that

$$\Delta = \begin{vmatrix} \cos(\alpha-\beta) & \cos(\beta-\gamma) & \cos(\gamma-\alpha) \\ \cos(\alpha+\beta) & \cos(\beta+\gamma) & \cos(\gamma+\alpha) \\ \sin(\alpha+\beta) & \sin(\beta+\gamma) & \sin(\gamma+\alpha) \end{vmatrix}$$

$$= -2\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$$

18. Show that

$$\Delta = \begin{vmatrix} \sin 3\alpha & \sin 2\alpha & \sin\alpha \\ \sin 3\beta & \sin 2\beta & \sin\beta \\ \sin 3\gamma & \sin 2\gamma & \sin\gamma \end{vmatrix}$$

$$= 2^6 \sin\alpha \sin\beta \sin\gamma \sin\frac{\alpha+\beta}{2} \sin\frac{\beta+\gamma}{2}$$

$$\times \sin\frac{\gamma+\alpha}{2} \sin\frac{\alpha-\beta}{2} \sin\frac{\beta-\gamma}{2} \sin\frac{\gamma-\alpha}{2}$$

19. By differentiation or otherwise, show that the determinant

$$\begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & a+x\sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & b+x\sin\beta \\ \sin(x+\gamma) & \cos(x+\beta) & c+x\sin\gamma \end{vmatrix}$$

is independent of x.

20. If $\Delta(x) = \begin{vmatrix} a_1+x & b_1+x & c_1+x \\ a_2+x & b_2+x & c_2+x \\ a_3+x & b_3+x & c_3+x \end{vmatrix}$

show that $\Delta''(x) = 0$ and that $\Delta(x) = \Delta(0) + Sx$, where S denotes the sum of all the cofactors of the elements in $\Delta(0)$.

21. Show that the maximum value of a third order determinant whose elements are 1 or – 1 is 4.

22. Without expanding, show that for all α,

$$\beta, \gamma, \delta;\ \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$$

23. Show that

$$\begin{vmatrix} a_1\ell_1+b_1m_1 & a_1\ell_2+b_1m_2 & a_1\ell_3+b_1m_3 \\ a_2\ell_1+b_2m_1 & a_2\ell_2+b_2m_2 & a_2\ell_3+b_2m_3 \\ a_3\ell_1+b_3m_1 & a_3\ell_3+b_3m_2 & a_3\ell_3+b_3m_3 \end{vmatrix}$$

$=0.$

24. If n is a positive integer, then find the value of

$$\begin{vmatrix} {}^{n+2}C_n & {}^{n+3}C_{n+1} & {}^{n+4}C_{n+2} \\ {}^{n+3}C_{n+1} & {}^{n+4}C_{n+2} & {}^{n+5}C_{n+3} \\ {}^{n+4}C_{n+2} & {}^{n+5}C_{n+3} & {}^{n+6}C_{n+4} \end{vmatrix}.$$

25. Prove that $\begin{vmatrix} \beta\gamma & \beta\gamma'+\beta'\gamma & \beta'\gamma' \\ \gamma\alpha & \gamma\alpha'+\gamma'\alpha & \gamma'\alpha' \\ \alpha\beta & \alpha\beta'+\alpha'\beta & \alpha'\beta' \end{vmatrix}$

$= (\alpha\beta'-\alpha'\beta)(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)$

A. Mutiple Choice Questions with ONE correct answer

1. If A and B are square matrices of equal degree, then which one is correct among the following? [IIT - 1995]
(A) $A + B = B + A$
(B) $A + B = A - B$
(C) $A - B = B - A$
(D) $AB = BA$

2 If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, then value of α for which $A^2 = B$, is - [IIT - 2003]
(A) 1
(B) –1
(C) 4
(D) no real values

3. If the system of equations $x + ay = 0$, $az + y = 0$, and $ax + z = 0$ has infinite solutions, then the value of a is - [IIT - 2003]
(A) –1
(B) 1
(C) 0
(D) no real values

4. Given
$2x - y + 2z = 2$, $x - 2y + z = -4$, $x + y + \lambda z = 4$ then the value of λ such that the given system of equation has no solution, is- [IIT - 2004]
(A) 3
(B) 1
(C) 0
(D) – 3

5. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$ then the value of α is [IIT - 2004]
(A) ± 1
(B) ± 2
(C) ± 3
(D) ± 5

6. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$A^{-1} = \left[\frac{1}{6}(A^2 + cA + dI)\right]$, then the value of c and d are [IIT - 2005]
(A) (–6, –11)
(B) (6, 11)
(C) (–6, 11)
(D) (6, –11)

7. If $P=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$ and $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $Q=PAP^T$ and $x=P^TQ^{2005}P$ then x is equal to **[IIT - 2005]**

(A) $\begin{bmatrix}1 & 2005\\ 0 & 1\end{bmatrix}$

(B) $\begin{bmatrix}4+2005\sqrt{3} & 6015\\ 2005 & 4-2005\sqrt{3}\end{bmatrix}$

(C) $\frac{1}{4}\begin{bmatrix}2+\sqrt{3} & 1\\ -1 & 2-\sqrt{3}\end{bmatrix}$

(D) $\frac{1}{4}\begin{bmatrix}2005 & 2-\sqrt{3}\\ 2+\sqrt{3} & 2005\end{bmatrix}$

8. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system A $\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$ has exactly two distinct solutions, is **[IIT - 2010]**

(A) 0 (B) 2^9-1
(C) 168 (D) 2

9. If matrix $A=\begin{bmatrix}a & b & c\\ b & c & a\\ c & a & b\end{bmatrix}$ where a, b, c are real positive numbers, abc = 1 and $A^TA = I$, then find the value of $a^3+b^3+c^3$. **[IIT - 2003]**

10. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form

$$\begin{bmatrix}1 & a & b\\ \omega & 1 & c\\ \omega & \omega & 1\end{bmatrix},$$

where each of a, b and c is either ω or ω^2. Then the number of distinct matrices in the set S is

(A) 2 (B) 6
(C) 4 (D) 8

[IIT - 2011]

11. Let $P=[a_{ij}]$ be a 3×3 matrix and let $Q=[b_{ij}]$, where $b_{ij}=2^{i+j}a_{ij}$ for $1\le i,j\le 3$. If the determinant of P is 2, then the determinant of the matrix Q is

(A) 2^{10} (B) 2^{11}
(C) 2^{12} (D) 2^{13}

[IIT - 2012]

B. Subjective Questions

12. If P is a 3×3 matrix such that $P^T=2P+I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix

$$X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\neq\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$ such that

(A) $PX=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$ (B) $PX=X$

(C) $PX=2X$ (D) $PX=-X$

[IIT - 2012]

13. If the adjoint of a 3×3 matrix P is $\begin{bmatrix}1 & 4 & 4\\ 2 & 1 & 7\\ 1 & 1 & 3\end{bmatrix}$, then the possible value(s) of the determinant of P is(are)

(A) -2 (B) -1
(C) 1 (D) 2

[IIT - 2012]

14. If M is 3×3 matrix, where det M = 1 and $MM^T = I$, where 'I' is an identity matrix, prove that det (M – I) = 0. **[IIT- 2004]**

Practice Problems [K]

1. Use Cramer's rule to solve the following systems.

   (i) $3x - 2y = 7i$
   $4x + y = 8$
   $3x + y - 7$

   (ii) $x - 2y + z = 3$
   $-x - y - z = 5$
   $z = -3$

2. Solve the system $\begin{cases} x - (\alpha - 1)y = 1, \\ \alpha x - 2y = 4 - \alpha, \ \alpha \in R \end{cases}$

3. Find the equation of the parabola $y = ax^2 + bx + c$ that passes through $(2, 4)$, $(-1, 1)$, and $(-2, 5)$.

4. Solve the system

$$\frac{1}{x} + \frac{1}{y} - \frac{1}{z} = \frac{1}{4}$$

$$\frac{2}{x} - \frac{1}{y} + \frac{3}{z} = \frac{9}{4}$$

$$-\frac{1}{x} - \frac{2}{y} + \frac{4}{z} = 1$$

5. For what value of k will the system $2x_1 + x_2 = 5$, $x_1 - 3x_2 = -1$, $3x_1 + 4x_2 = k$ be solvable? Solve the system for that value of k.

6. For what of k the following system has non-trivial solution and find the solution for $k = -1$.
   $x + 2y + 3z = kx$, $\quad 2x + y + 3z = ky$,
   $2x + 3y + z = kz$.